MATH-111(en)
Linear Algebra

FALL 2025
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Graded Homework 2

3 November 2025

Rules:

- Keep your solution concise but complete (i.e. justify your steps).
- Submit (either LaTeX or scan/photo of legible handwriting) on Moodle by November 14.
- Aim to spend 30 minutes in total.
- Address questions to your graders: Eliota Braha and Omar Zakariya.

Problem 1

Let $A \in \mathbb{R}^{m\times n}$ be a matrix such that its reduced echelon form has exactly k zero rows. Determine the dimension of $\mathrm{Ker}(A)$ and of $\mathrm{Col}(A)$ in terms of m, n and k.

Problem 2

Let $f : \mathbb{P}_3 \to \mathbb{R}^4$ be the linear function defined by

$$f(1) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad f(x) = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \quad f(x^2) = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 3 \end{pmatrix}, \quad f(x^3) = \begin{pmatrix} -3 \\ 5 \\ 3 \\ 3 \end{pmatrix}$$

(a) Find a basis for $\mathrm{Ker}(f)$.

(b) Determine whether $\begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}$ is in $\mathrm{Im}(f)$.

BE AWARE : "Grade" just means that we provide you with feedback / corrections on your submission. It does <u>not</u> mean that you will receive a numeric grade. This is an exercise and does not count towards the final grade of this course. Participation is <u>not</u> mandatory.

For better feedback, we will assign letters to each graded problem:
A = *good solution, only minor mistakes or imperfect (but still clear) notation.*
B = *your solution catches relevant aspects but also has considerable flaws or gaps.*
C = *your solution was mostly wrong and/or there were many substantial mistakes or gaps.*

1) $\operatorname{Ker}(A) = \operatorname{Ker}(REF(A))$ so if $REF(A)$ has exactly k zero rows, then it has $m-k$ pivot rows, hence $m-k$ pivot columns, hence $n-m+k$ free columns; since each free columns gives a basis vector for the solution space of $Ax=0$, i.e. for $\operatorname{Ker}(A)$, then $\dim \operatorname{Ker}(A) = n-m+k$

$$\begin{array}{c} m-k\left\{ \begin{array}{c} \\ \\ \\ \end{array}\right. \\ k\left\{ \begin{array}{c} \\ \\ \end{array}\right. \end{array} \begin{pmatrix} \boxed{1} & * & 0 & * & 0 & * \\ 0 & 0 & \boxed{1} & * & 0 & * \\ 0 & 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

While $\operatorname{Col}(A) \neq \operatorname{Col}(REF(A))$, they have the same dimension since any linear relation between the columns of A gives rise to a linear relation between the columns of $REF(A)$

so $\operatorname{rank}(A) = \dim(\operatorname{Col}(A)) = \#\text{pivots} = m-k$.

2) With respect to the basis $\{1, x, x^2, x^3\}$ of $\mathbb{P}_3$, f is represented by the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & -3 \\ -1 & 2 & 1 & 5 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 3 \end{pmatrix} \xrightarrow[\text{add } -R1 \text{ to } R4]{\substack{\text{add } R1 \text{ to } R2 \\ \text{add } -R1 \text{ to } R3}} \begin{pmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 3 & 6 \\ 0 & 3 & 3 & 6 \end{pmatrix} \xrightarrow{\text{add } -R3 \text{ to } R4} \begin{pmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{add } -3R2 \text{ to } R3} \begin{pmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{add } R2 \text{ to } R1} \begin{pmatrix} \boxed{1} & 0 & 1 & -1 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = REF(A).$$

a) $$\operatorname{Ker}(A) = \operatorname{Ker}(REF(A)) = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \;\middle|\; \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \;\middle|\; \begin{array}{l} x = -z + t \\ y = -z - 2t \end{array} \right\}$$

$$= \left\{ z \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; z, t \in \mathbb{R} \right\}$$ $\Rightarrow \operatorname{Ker}(A)$ has a basis given by $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$

$\Downarrow$

$\operatorname{Ker}(f)$ has a basis given by $\{-1 - x + x^2,\ 1 - 2x + x^3\}$

b) $\operatorname{Im}(f) = \operatorname{Col}(A)$ is spanned by the pivot columns of A, i.e. $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ 2 \\ 2 \end{pmatrix}$

However, $\begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix} \neq \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 2 \\ 2 \end{pmatrix}$ because $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ would force $\alpha = \beta = 0$ since $\det \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = 3 \neq 0$.